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DETERMINATION OF THE DYNAMIC SHEAR MODULUS AND THE DEPTH OF THE DOMINANT LAYER OF A VIBRATING ELASTIC MEDIUM

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Abstract—It is shown that errors up to 20 per cent can arise in the estimation of the dynamic shear modulus of the foundation under a vibrating structure if the foundation, as hitherto generally assumed, is regarded as an isotropic half-space instead of an isotropic stratum on a relatively rigid bed which, in many practical cases, is a more realistic model. The problem then reduces to the determination of the depth and the shear modulus of some layer immediately under the structure, for this governs more the response of the structure than does the average modulus of the entire medium.

The present work, based on recent results of the torsional vibration of a rigid circular body on an elastic stratum, corrects these errors by first establishing in a novel manner the depth of this dominant layer and then determining its shear modulus. Test procedure and the choice, with magnitude, of parameters to be used in order to obtain reliable results are given. The work also gives the method of detecting cases in which the vibrating medium cannot be replaced by the theoretical model on which it is based.

1. INTRODUCTION

IN THIS age of rocketry and heavy machinery, when the soil is continually subjected to dynamic loading, a realistic evaluation of the soil resistance based on its dynamic elastic properties is essential. The successful launching of space-crafts and the safety of such large structures as buildings, dams and bridges vibrating on their foundations require some accurate knowledge of the elastic constants. It has been recognized for a long time that such an evaluation of soil constants has to be carried out *in situ* since static laboratory tests as in triaxial loading of a soil sample gives a much lower value of the elastic modulus than under dynamic conditions. The change in soil compaction is not a major cause of this difference because an *in situ* static test would also give a much lower value than the dynamic modulus.

One method of determining the dynamic modulus of elasticity, usually employed by earthquake engineers, is based on a measurement of the velocities of waves of dilatation (Push Waves) and of distortion (Shear Waves). The source of wave propagation is either a detonated seismic charge or the impact of a sledge-hammer on a metal plate. The pulse is received by a geophone connected to a seismograph, essentially a timing unit, and placed at a known distance from the wave source. This method has the disadvantage that it requires a knowledge of the Poisson's ratio of the soil because, in practice, only the arrival of the first pulse corresponding to compressional or push waves is clearly distinct. The slower waves—shear waves and Rayleigh surface waves—are generally indistinguishable. Seismologists, therefore, generally assume a value for Poisson's ratio. Also, this method is insensitive to the dependence of the modulus on the strain imposed on the soil since only very small strains are involved in the procedure unlike those imposed on soils under large vibrating structures.

Resonance techniques based on the vertical vibrations of a rigid circular body excited by an electromagnetic vibrator are now being accepted as a very convenient procedure of *in situ* experiments as shown in references cited by Grootenhuis and Awojobi [1], who have used this method to determine the dynamic elastic constants of London Clay on a site at Imperial College Field Station, near Ascot. However, the theory for analysing results is difficult and, even on an assumption of an isotropic half-space, only approximate solutions are known. Also, the problem unavoidably depends on Poisson's ratio of the soil as in the case of measuring dilatational wave velocity.

It was first recognised by Reissner and Sagoci [2] that the torsional oscillation of a rigid circular body on an elastic half-space can be used to determine the shear modulus of the medium. The significant advantage of this method is that the mode of vibration of particles in the medium is such that only shear waves are propagated and, therefore, the theory is completely independent of Poisson's ratio. Unfortunately, the authors' proposals of determining shear modulus based on the measurement of the phase lag between the in-phase and quadrature components of stress cannot be expected to give accurate results. Indeed, as it has been discussed by Grootenhuis and Awojobi [1], any measurement that involves "dispersion damping", that is, energy dissipation due to the dispersion of waves in the medium, is unreliable because 'thermal damping', inherent in the medium by virtue of friction due to relative motion between its particles, is not taken into account. Thus, phase angle or amplitude of displacement should not be used directly in calculating results except as a means of detecting resonant frequency which is much less dependent on damping than either of the above parameters. Robertson [3], without these pertinent comments, has rightly suggested the use of resonant frequency in determining shear modulus using his theory for torsional oscillation of a rigid circular body on an elastic half-space.

The present work breaks through one of the great setbacks that resonance techniques suffer—the assumption that the soil has constant elastic modulus and is uniformly composed to an infinite depth. Reliance on this simplifying assumption has hitherto been due to lack of an adequate theory for the mixed boundary-value problem of vibration of rigid bodies on an elastic stratum. In a most recent work [4], the author has proposed a theory for the torsional oscillation of a rigid circular body on an infinite elastic stratum. The theory shows good agreement with published experimental data and reduces to the asymptotic case of a semi-infinite medium. The theory can now be reliably used to determine the depth and the shear modulus of the top or dominant layer of a semi-infinite medium which, otherwise, would have been assumed to be of uniform composition. Since the depth is also given referred to the base radius of the rigid circular body as unit, this new method conversely establishes implicitly the range of base radius that should be used, in a case where the tester has some idea of the depth of the top soil, in order that the dominant stratum might be a fair approximation to a semi-infinite medium. The work also defines the range of inertia ratios of the rigid body for which resonant frequency is substantially sensitive to changes in stratum depth. Finally, from these factors and the experience of the author in previous field tests [1] recommendations are made on choice of parameter sizes and on test procedure.

2. GOVERNING EQUATIONS

The torsional vibration of a rigid circular body on an infinite elastic stratum and subjected to a sinusoidal torque has been exactly reformulated by the author in the work [4] already cited in terms of the dual integral equations:

$$\int_0^\infty \frac{\tanh(\alpha_2\tilde{h})}{\alpha_2} F(\eta)J_1(\eta\tilde{r})\,d\eta = c\tilde{r} \quad (0 < \tilde{r} < 1)$$
$$\int_0^\infty F(\eta)J_1(\eta\tilde{r})\,d\eta = 0 \quad (\tilde{r} > 1) \tag{1}$$

which, at present, cannot be solved exactly and in which the notation of the previous work is retained.

By using the new approximation $\tanh x \simeq x/(1+x^2)^{1/2}$ for all real values of x, and substituting in the above it has been found that a stratum excited at a frequency factor η_2 behaves approximately as a semi-infinite medium excited at an equivalent frequency factor η_{2e} related to the stratum depth, $\tilde{h}$ referred to the base radius, by the equation

$$\eta_{2e} = (\eta_2^2 - 1/\tilde{h}^2)^{1/2}. \tag{2}$$

It is ultimately established that the torsional displacement amplitude of the rigid body can be expressed as

$$|\hat{\theta}| = 1/\sqrt{(P^2+Q^2)} \tag{3}$$

where

$$P = \tfrac{16}{3} - (\tilde{J}\eta_2^2 + \tfrac{8}{45}\eta_{2e}^2) - 0{\cdot}89\eta_{2e}^2(1 - 0{\cdot}63\eta_{2e}^2 + 0{\cdot}305\eta_{2e}^4 - 0{\cdot}083\eta_{2e}^6 + 0{\cdot}0122\eta_{2e}^8)$$

and

$$Q = 0{\cdot}755\eta_{2e}^3(1 - 0{\cdot}56\eta_{2e}^2 + 0{\cdot}216\eta_{2e}^4 - 0{\cdot}040\eta_{2e}^6 + 0{\cdot}0027\eta_{2e}^8)$$

where $\tilde{J} = J/\rho R^5$ is the non-dimensional polar inertia of the body on a medium of density ρ.

3. ERRORS IN SHEAR MODULUS DUE TO ASSUMPTION OF A SEMI-INFINITE MEDIUM

Equation (3) has been used to generate the response curves shown in Fig. 1. Inertia ratios $\tilde{J}$ from 1 to 40 on media of non-dimensional depths $\tilde{h} = 1, 2, 3, 4, 5$ and ∞ are used and these are sufficient for most practical cases. The same curve is used for more than one value of $\tilde{h}$ if the difference is so small that graphical representation on the chosen scale cannot reasonably justify their separation. The form of these curves has already been discussed in the work cited and shall not be repeated here.

It is pertinent to recall now that shear modulus, G is related to frequency factor, η_2 by the expression $\eta_2 = R\Omega/c_2$ where Ω is frequency of excitation and c_2, the velocity of waves of distortion, is given by

$$c_2 = \sqrt{\frac{G}{\rho}}.$$

Thus, by measuring resonant frequency experimentally shear modulus will be calculated from

$$G = \frac{\rho R^2\Omega^2}{\eta_2^2}. \tag{4}$$

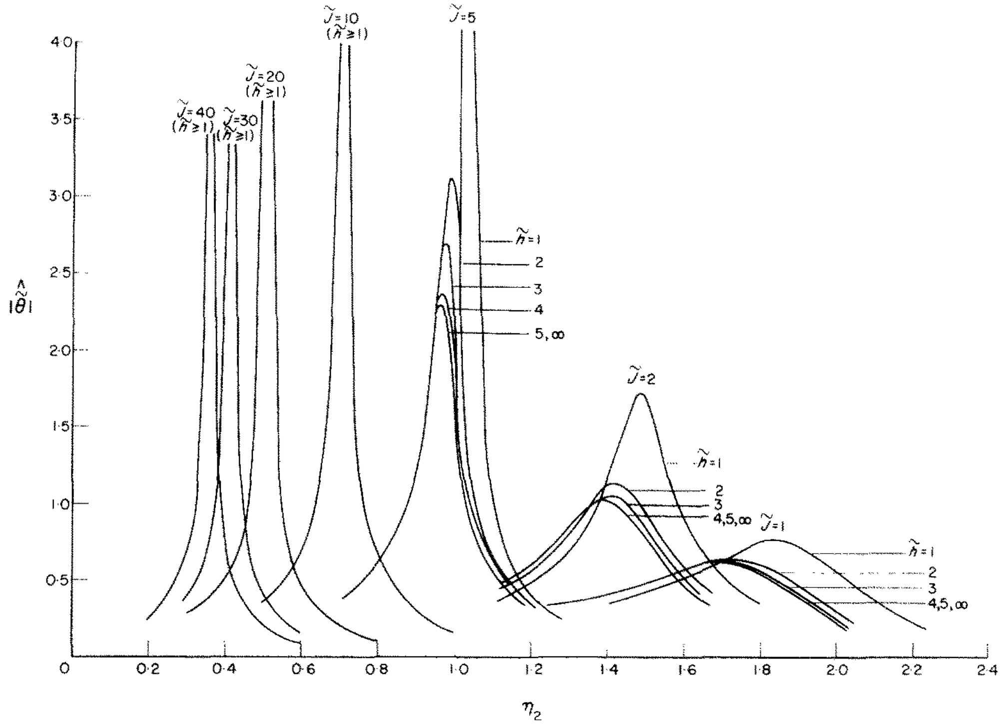

FIG. 1. Resonance curves for rigid circular bodies of different inertia ratios, $\tilde{J}$, on strata of different depths, $\tilde{h}$.

If a semi-infinite medium is assumed as, for example by Robertson [3] then the resonant frequency factor, η_2, to be used in equation (4) will correspond to the peak of the curve of $\tilde{h} = \infty$ for the chosen inertia ratio $\tilde{J}$. Equation (4) shows that G is inversely proportional to the square of frequency factor thereby emphasizing the need to use the correct curve of non-dimensional depth especially for low inertia ratio bodies. Thus, in the case of $\tilde{J} = 1$, if the stratum depth is actually $\tilde{h} = 1$ with $\eta_2 = 1{\cdot}83$, then the value of G, calculated on an assumption of a semi-infinite medium with $\eta_2 = 1{\cdot}67$ at resonance, is readily shown to be greater than, and with 20% in error of, the correct value of G.

This section, therefore, establishes the fact that if all that is required is the shear modulus, then the resonant frequency factor of high inertia ratio bodies ($\tilde{J} \geq 40$) can be used with negligible error for all values of stratum depth greater than the base radius. The main source of error in this case may arise from the practical measurement of the resonant frequency because large bodies tend to produce large strains from which non-linear effects would change the soil stiffness and, consequently, resonant frequency. This is overcome by the method of measurement discussed later.

4. DETERMINATION OF THE DOMINANT LAYER DEPTH

In many large structures, it is usual to find that the soil changes in form before a depth comparable with the base dimensions is reached. The response of such structures to exciting forces may critically depend on determining the depth of the top soil since this provides a great part of the resistance the soil offers to disturbing forces on the structure.

All that is required is to measure the resonant frequencies Ω_a and Ω_b corresponding to two inertia ratios of the rigid body. Assuming that the base radius is unchanged and that the shear modulus is independent of frequency then, from equation (4)

$$\frac{\Omega_a}{\Omega_b} = \frac{\eta_{2_a}}{\eta_{2_b}}. \tag{5}$$

Thus, if for any suitable pair of inertia ratios a plot of η_{2_a}/η_{2_b} at resonance against $\tilde{h}$ is made, then the appropriate $\tilde{h}$ can be found in a given experiment in which the left-hand side of equation (4) has been determined by measuring Ω_a and Ω_b. Such a family of curves is given in Fig. 2 where resonant frequency factors η_{2_a} and η_{2_b} have been derived from enlarged portions of Fig. 1 in the vicinity of the peaks. The pairs of inertia ratios have been suitably

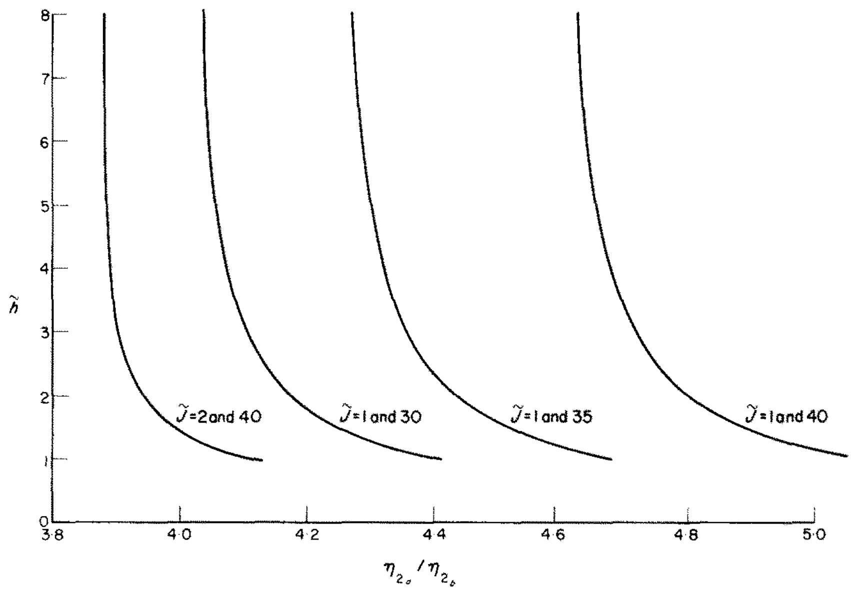

FIG. 2. Curves for the determination of the unknown depth, $\tilde{h}$, of the dominant layer.

chosen as extremes of heavy and light rigid bodies to ensure sensitive dependence on $\tilde{h}$.

It is useful to note that the resonant frequency factor for high inertia ratios ($\tilde{J} \geq 40$) in which the depth $\tilde{h}$ is such that $0 \leq \eta_{2e} \leq 0{\cdot}5$ can be calculated within 1% error on the assumption that the expression P in equation (3) vanishes. Furthermore, in this range of η_{2e} we may neglect terms of order η_{2e}^4.

We, therefore, readily find that for high inertia ratio bodies

$$\tfrac{16}{3} - (\tilde{J}\eta_2^2 + \tfrac{8}{45}\eta_{2e}^2) - 0{\cdot}89\eta_{2e}^2 \simeq 0$$

or

$$\eta_2^2 \simeq \left(5{\cdot}33 + \frac{1{\cdot}07}{\tilde{h}^2}\right) \Big/ (\tilde{J} + 1.07) \tag{6}$$

where use has been made of equation (2) in eliminating η_{2e}^2. Equation (6) reduces, in the case of the semi-infinite medium to

$$\eta_2^2 \simeq \frac{5 \cdot 33}{\bar{J} + 1 \cdot 07} \tag{7}$$

which is used in Table 1 to compare with the results of Robertson [3] as shown in Table 11 of his work. The Table shows that equation (7) can be used with negligible error for all $\bar{J}$ greater than 40.

TABLE 1.

$\bar{J}$	Resonant frequency factor, η_2	
	(Robertson)	(Awojobi)
20	0·505	0·502
30	0·415	0·414
40	0·361	0·361
50	0·323	0·323

It should also be noted that when the measured ratio lies outside the curve, the experiment can be repeated with a suitable change of base radius to ensure that the point lies within the range of the curve.

It is also important to note that it has been assumed in the above that the medium behaves as the equivalent of an infinite elastic stratum of constant shear modulus and resting on a rigid foundation. The argument has been set out in the previous work [4] that this assumption is not unreasonable in many practical cases. However, where the medium behaves otherwise, the depth determined above will not be of any practical use since the performance of rigid bodies of other inertia ratios cannot be predicted from the depth and shear modulus consequently determined.

In order to ascertain that the model is useful, it is necessary to carry out the test for other pairs, or at least one more pair, of inertia ratios. If the values of $\tilde{h}$ obtained in the several cases agree within reasonable limits then the medium is behaving as the above theoretical model. Also, the curves in Fig. 1 show that, provided $\tilde{h} > 5$, the resonant frequency factor for a given inertia ratio does not sensitively depend on further increase of $\tilde{h}$. Therefore, even if the values of $\tilde{h}$ for several pairs of inertia ratio do not agree but are *all* greater than 5, we can conclude that the medium is uniformly composed or equivalent to a uniformly composed stratum of depth $\tilde{h}$ of *at least* 5 since all the several $\tilde{h}$ would give approximately same shear modulus so that the performance of any other rigid body can be predicted within reasonable limits.

5. DETERMINATION OF THE SHEAR MODULUS

Now that the depth $\tilde{h}$ has been determined, we can now revert to Fig. 1 and using the appropriate $\tilde{h}$ curve we determine the resonant frequency factor η_2 and calculate G using equation (4).

It is desirable, for accurate results, to use equation (3) at very small intervals of η_2 to obtain the resonance curve in the neighbourhood of the peak to establish as accurately as possible the value of resonant frequency factor.

6. TEST PROCEDURE FOR ACCURATE MEASUREMENT OF RESONANT FREQUENCY

Equation (4) shows that just as it is important to use the appropriate theoretical resonant frequency factor, η_2 so also it is important to measure as accurately as possible, the resonant frequency Ω especially when G depends on the square of each of these quantities. The experience of the author as recorded in the work cited [1] shows that soil dynamic properties are sensitive to strain amplitudes so that any measurement of resonant frequency based on varying the amplitude cannot be reliable especially when, by this method, resonance occurs at maximum amplitude. Also, this method cannot establish whether the range of amplitude encountered especially in the vicinity of resonance is not beyond the elastic range.

The above considerations led the author to appreciate in [1] the importance of the following alternative method of measuring resonant frequency of any system which may exhibit non-linearity: the amplitude should be set to a small constant value whilst the exciting force is changed; resonant frequency will then correspond to the position of minimum exciting force. The onset of non-linearity can be detected by increasing the set amplitude. At low amplitudes, that is, within the elastic range the minimum force will occur at the same frequency. As the set amplitude is gradually increased a shift of the minima to the left (indicating a softening of the ground or reduced stiffness) or, as may rarely happen, to the right (indicating increased stiffness) clearly establishes the onset of nonlinearity.

7. CONCLUSIONS

The dynamic shear modulus of an elastic medium can now be determined without assuming that the medium is a semi-infinite isotropic medium. Rather, a novel method of determining the depth of the most active or top layer of soil immediately under a vibrating structure has been established so that the shear modulus of this active layer is more accurately found. Therefore, the response of the structure which depends more on this active layer than on the entire medium can be better predicted than hitherto possible.

It is emphasized that resonant frequency is the most convenient parameter to measure in order to achieve accurate results as this is not greatly affected by thermal damping which no theory takes into account. It is important to measure resonant frequency accurately and a procedure to ensure the elimination of non-linear effects on resonant frequency, invariably exhibited by soil, has been discussed. It has also been remarked that this resonance technique of finding dynamic shear modulus based on torsional vibration of the medium has a great advantage over all other existing methods because it is independent of the Poisson's ratio of the medium, another unknown elastic constant.

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Абстракт—Показано, что погрешности могут достигнуть 20% при определении динамиуеского модуля сдвига основания под колебательной системой, если основание, как подразумивается до сих пор, рассматривается в виде изотропного полупространства вместо изотропного слоя, на относительно жестком основании. Новой тип основания, для большинства практиуеских случаев, оказывается более реальной моделью. Задауа сводится к определению глубины и модуля сдвига некоторого слоя, непосрелственно под системой, поетому, что зтие прелположения определяют более поведение системы, уем зто происходит в случае пользования среднем модулем всей среды.

Предлагается работа, основанная на последних результатах исследования крутильных колебаний жесткого круглого тела на упругом основании, поправляет зти погрешности во первых путем введения учета, новым способм, толшины преобладающего слоя, и далее, путем определения модуля сдвига. Даются схема испытания и Выбор веничины использованных параметров, с целью получения надежных результатов. Работа дает также метод обнаружения слуаев, в которых колебательная среда не может быть заменена теоретиуеской моделвю, на которой она основана.